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## On A Simple Construction Of Triangle Centers X(8), X(197), X(K) & X(594)

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### Abstract

In this article, we provide a new method for constructing the Nagel Point {X(8)} , Cevian quotient of Symmedian Point, Nagel Point {X(197)} and the isogonal conjugate of 1<sup>st</sup> Hatzipolakis-Yiu Point {X(594)}. In addition, we also establish some collinearity and concurrence.

**Keywords:** triangle centers, Nagel point, Carnot's theorem, collinearity and concurrence.

### Introduction

In the literature of Encyclopedia of Triangle Centers [1], there is a list of over 2000 triangle centers. Among those X(8), X(197) , X(K) and X(594) are four such triangle centers. In this note, we devote our study for the construction of these points and their related coincidence.

In article [2], *Larry Hoehn* gives another way to construct the Nagel point using only the incircle and not the excircles. In this article we deal with a simple and an elegant construction of these four points which reveals a new characterization of X(8), X(197), X(K), X(594) and their unexpected coincidence.

### Terminology

Triangle center	Its name
X(8)	Nagel Point
X(197)	Cevian Quotient of Symmedian Point, Nagel Point
X(K)	Isogonal Conjugate of X(5019), Isotomic Conjugate of X(940), The Polar Conjugate of X(4185), The Trilinear Pole Of Line X(523)X(4391)
X(594)	Isogonal Conjugate of 1st Hatzipolakis-Yiu Point

### Construction

Given a triangle  $ABC$ , let  $D, E, F$  be the midpoints of the sides  $BC, CA$  and  $AB$ , construct a circle  $O_A$  with  $A$  as center and  $AD$  as radius which intersects the line drawn through  $D$  and parallel to the internal angular bisector of angle  $A$  at  $L_A$  (it is clear that the point  $L_A$  is the reflection of  $D$  with respect to the external angular bisector of angle  $A$ ). Similarly, define the points  $L_B$  and  $L_C$ . Consider the points  $N_A, N_B$  and  $N_C$  as

$$N_A = BL_C \cap CL_B, N_B = AL_C \cap CL_A \text{ and } N_C = BL_A \cap AL_B \text{ then}$$

(I) The line segments  $AN_A, BN_B$  and  $CN_C$  concur at  $X(594)$

(II) The line segments  $L_A N_A, L_B N_B$  and  $L_C N_C$  concur at  $X(8)$

Consider the points  $N'_A, N'_B, N'_C, L'_A, L'_B, L'_C, M'_A, M'_B$  and  $M'_C$  as  $N'_A = AN_A \cap BC$ ,  $N'_B = BN_B \cap CA$ ,  $N'_C = CN_C \cap AB$ ,  $L'_A = AX(8) \cap BC$ ,  $L'_B = BX(8) \cap CA$ ,  $L'_C = CX(8) \cap AB$ ,  $M'_A = L_A N_A \cap BC$ ,  $M'_B = L_B N_B \cap CA$  and  $M'_C = L_C N_C \cap AB$  then

**(III) The line segments  $AM'_A, BM'_B$  and  $CM'_C$  are concurrent at  $X(K)$**

**(IV) The lines  $N_A L'_A, N_B L'_B$  and  $N_C L'_C$  are concurrent at  $X(197)$**

**(V) There exists a conic through any two triads out of three triads  $(D, E, F)$ ,**

**$(N'_A, N'_B, N'_C)$  and  $(L'_A, L'_B, L'_C)$ .**

In order to prove (1), (II), (III), (IV) and (V), we will make use of barycentric coordinates. If a triangle ABC has side lengths  $BC = a$ ,  $CA = b$ ,  $AB = c$  then  $A = (1 : 0 : 0)$ ,  $B = (0 : 1 : 0)$ ,  $C = (0 : 0 : 1)$ ,  $D = (0 : 1 : 1)$ ,  $E = (1 : 0 : 1)$  and  $F = (1 : 1 : 0)$  in homogeneous barycentric coordinates with reference to ABC [4].

### Proposition 1

The equation of line joining of two points with coordinates  $(x_1 : y_1 : z_1)$  and

$$(x_2 : y_2 : z_2) \text{ is } \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \text{ or } x(y_1 z_2 - y_2 z_1) + y(z_1 x_2 - z_2 x_1) + z(x_1 y_2 - x_2 y_1) = 0$$

### Proposition 2

The intersection of the two lines  $p_1 x + q_1 y + r_1 z = 0$  and  $p_2 x + q_2 y + r_2 z = 0$  is the point  $(q_1 r_2 - q_2 r_1 : r_1 p_2 - r_2 p_1 : p_1 q_2 - p_2 q_1)$ .

### Proposition 3

Three lines  $p_i x + q_i y + r_i z = 0$ ,  $i = 1, 2, 3$ , are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

### Proposition 4

The barycentric coordinate of the points which are the reflection of an arbitrary point  $P(u : v : w)$  with respect to the external angular bisector of angle A is

$$P_A = (c^2 v + b^2 w + b c (u + v + w) : -b^2 w : -c^2 v),$$

with respect to the external angular bisector of angle B is

$$P_B = (-a^2 w : c^2 u + a^2 w + a c (u + v + w) : -c^2 u),$$

and with respect to the external angular bisector of angle C is

$$P_C = (-a^2 v : -b^2 u : b^2 u + a^2 v + a b (u + v + w)) \text{ [4].}$$

### Corollary

By replacing  $u = 0$ ,  $v = w = 1$  then  $P_A = L_A = ((b+c)^2 : -b^2 : -c^2)$

By replacing  $v = 0$ ,  $u = w = 1$  then  $P_B = L_B = (-a^2 : (a+c)^2 : -c^2)$

By replacing  $w = 0$ ,  $u = v = 1$  then  $P_C = L_C = (-a^2 : -b^2 : (a+b)^2)$

### Note

1. The triangles  $ABC$  and  $P_AP_BP_C$  are perspective with perpector of the isogonal conjugate of  $P$  [4].
2. The excentral triangle and  $P_AP_BP_C$  are perspective if and only if  $P$  lies on the Neuberg cubic of excentral triangle [4].

Using the prepositions listed above we list out the barycentric coordinates of the specified points and equation of the lines in barycentric system (we use standard notations such as  $a = BC$ ,  $b = CA$ ,  $c = AB$  and  $s =$  semi perimeter,  $R =$  circum radius,  $r =$  inradius,  $\Delta =$  area).

**Table 1.** Point and its Barycentric Coordinates

Point	barycentric coordinates	Point	barycentric coordinates
$L_A$	$((b+c)^2 : -b^2 : -c^2)$	$L'_A$	$(0 : (s-b) : (s-c))$
$L_B$	$(-a^2 : (a+c)^2 : -c^2)$	$L'_B$	$((s-a) : 0 : (s-c))$
$L_C$	$(-a^2 : -b^2 : (a+b)^2)$	$L'_C$	$((s-a) : (s-b) : 0)$
$N_A$	$(-a^2 : (c+a)^2 : (a+b)^2)$	$N'_A$	$(0 : -(c+a)^2 : -(a+b)^2)$
$N_B$	$((b+c)^2 : -b^2 : (a+b)^2)$	$N'_B$	$(-(b+c)^2 : 0 : -(a+b)^2)$
$N_C$	$((b+c)^2 : (c+a)^2 : -c^2)$	$N'_C$	$(-(c+b)^2 : -(c+a)^2 : 0)$

Point	barycentric coordinates
$M'_A$	$(0 : sc^2+abc : sb^2+abc) = (0 : c^2+4Rr : b^2+4Rr)$
$M'_B$	$(c^2+4Rr : 0 : a^2+4Rr)$
$M'_C$	$(b^2+4Rr : a^2+4Rr : 0)$

Point	barycentric coordinates
$X(8)$	$((s-a) : (s-b) : (s-c)) = (\cot A/2 : \cot B/2 : \cot C/2)$
$X(197)$	$(a^2 [s^2 (b^2 + c^2 - a^2) - s (bc(b+c) + ca(c-a) + ab(b-a) + abc(b+c-a))] : b^2 [s^2 (c^2 + a^2 - b^2) - s (ca(c+a) + ab(a-b) + bc(c-b) + abc(c+a-b))] : c^2 [s^2 (a^2 + b^2 - c^2) - s (ab(a+b) + bc(b-c) + ca(a-c) + abc(a+b-c))])$ $= af(A,B,C) : bf(B,C,A) : cf(C,A,B)$ <p>Where <math>f(A,B,C) = a[-a^2 \tan A/2 + b^2 \tan B/2 + c^2 \tan C/2]</math></p>
$X(594)$	$((c+b)^2 : (c+a)^2 : (a+b)^2)$
$X(K)$	$((b^2+4Rr)(c^2+4Rr) : (c^2+4Rr)(a^2+4Rr) : (a^2+4Rr)(b^2+4Rr))$

**Table 2.** Line and its Barycentric Equation

Line	Equation	Line	Equation	Line	Equation
$AL_B$	$yc^2 + z(c+a)^2 = 0$	$BL_A$	$xc^2 + z(b+c)^2 = 0$	$CL_B$	$x(c+a)^2 + ya^2 = 0$
$AL_C$	$y(a+b)^2 + zb^2 = 0$	$BL_C$	$x(a+b)^2 + za^2 = 0$	$CL_A$	$xb^2 + y(b+c)^2 = 0$

Line	Equation
$AN_A$	$y(a+b)^2 - z(c+a)^2 = 0$
$BN_B$	$x(a+b)^2 - z(b+c)^2 = 0$
$CN_C$	$x(c+a)^2 - y(b+c)^2 = 0$

Line	Equation
$AX(8)$	$y(s-c) - z(s-b) = 0$
$BX(8)$	$x(s-c) - z(s-a) = 0$
$CX(8)$	$x(s-b) - y(s-a) = 0$

Line	Equation
$L_A N_A$	$xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0$
$L_B N_B$	$yca(c-a) - s[c^2(y+z) - a^2(y+x)] - abc(z-x) = 0$
$L_C N_C$	$zab(a-b) - s[a^2(z+x) - b^2(z+y)] - abc(x-y) = 0$

Line	Equation
$AM'_A$	$y(b^2+4Rr) - z(c^2+4Rr) = 0$
$BM'_B$	$x(a^2+4Rr) - z(c^2+4Rr) = 0$
$CM'_C$	$x(a^2+4Rr) - y(b^2+4Rr) = 0$

Line	Equation
$N_A L'_A$	$a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0$
$N_B L'_B$	$b^2[z(s-a)-x(s-c)] + y(c-a)[s(c+a)-ca] = 0$
$N_C L'_C$	$c^2[x(s-b)-y(s-a)] + z(a-b)[s(a+b)-ab] = 0$

### Theorem-1

*The line segments  $AN_A$ ,  $BN_B$  and  $CN_C$  are concur at  $X(594)$ .*

#### Proof

Using table 2, it is easy to check that the point  $X(594)\{((c+b)^2 : (c+a)^2 : (a+b)^2)\}$  satisfies the lines  $AN_A \{y(a+b)^2 - z(c+a)^2 = 0\}$ ,  $BN_B \{x(a+b)^2 - z(b+c)^2 = 0\}$  and  $CN_C \{x(c+a)^2 - y(b+c)^2 = 0\}$ .

Hence the lines  $AN_A$ ,  $BN_B$  and  $CN_C$  are concurrent at  $X(594)$ . (see figure -1)

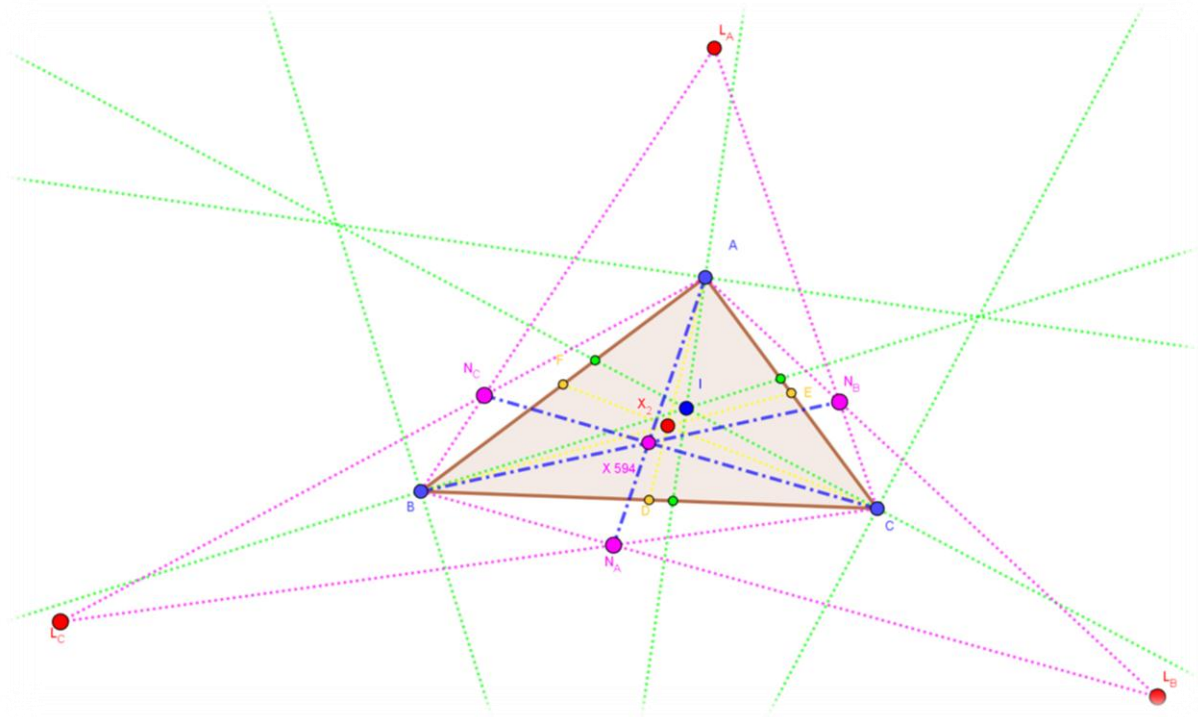


Figure 1.

### Theorem-2

*The line segments  $L_A N_A$ ,  $L_B N_B$  and  $L_C N_C$  are concurrent at  $X(8)$ .*

### Proof

Using table 2, it is easy to check that the point  $X(8)\{((s-a) : (s-b) : (s-c))\}$  satisfies the lines  $L_A N_A$   $\{ xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0 \}$ ,  $L_B N_B$   $\{ yca(c-a) - s[c^2(y+z) - a^2(y+x)] - abc(z-x) = 0 \}$  and  $L_C N_C$   $\{ zab(a-b) - s[a^2(z+x) - b^2(z+y)] - abc(x-y) = 0 \}$ .

For example

Consider  $xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0 \dots\dots(1)$

Replace  $x = (s-a)$ ,  $y = (s-b)$ ,  $z = (s-c)$ ,  $x+y = c$ ,  $x+z = b$  and  $y-z = c-b$

$$\begin{aligned} xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) &= bc(s-a)(b-c) - s[b^2c - c^2b] - abc(c-b) \\ &= bc(s-a)(b-c) - bc(s-a)(b-c) = 0 \end{aligned}$$

Hence (1) is satisfied by the point  $X(8)\{((s-a) : (s-b) : (s-c))\}$

That is the line  $L_A N_A$  contains  $X(8)$ .

In the similar manner we can verify that  $L_B N_B$  and  $L_C N_C$  also contains  $X(8)$ .

So the lines  $L_A N_A$ ,  $L_B N_B$  and  $L_C N_C$  are concurrent at **X(8)**. (see figure -2)

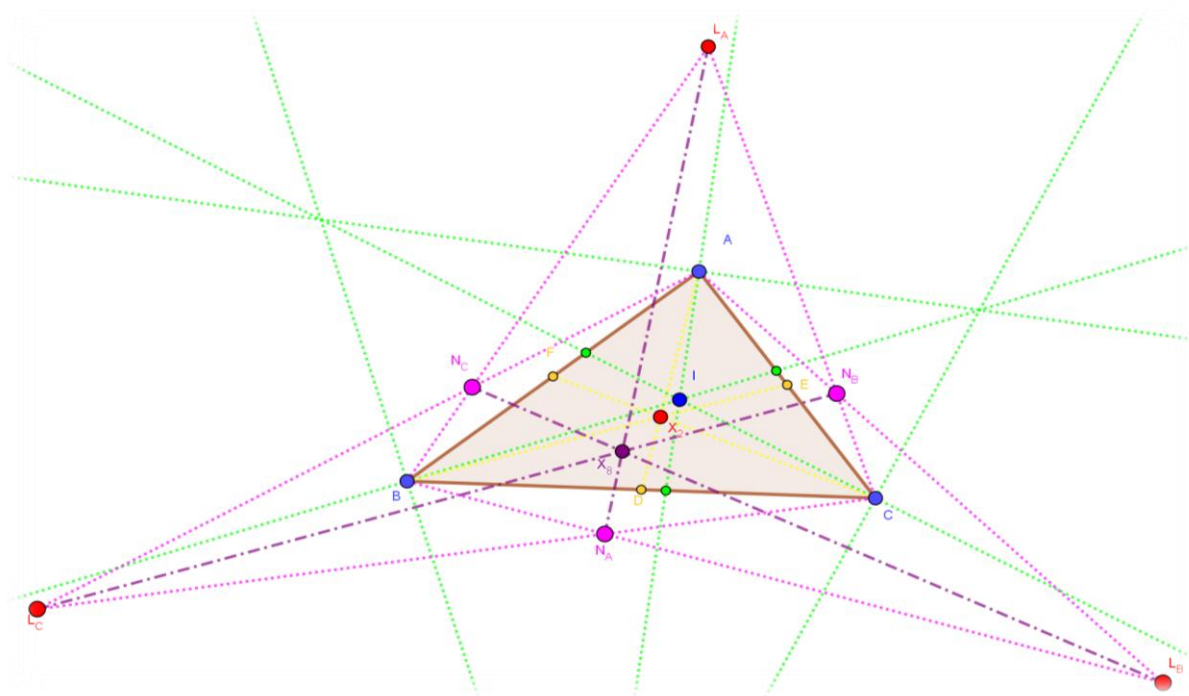


Figure 2.

### Theorem – 3

The line segments  $AM'_A$ ,  $BM'_B$  and  $CM'_C$  are concur at  $X(K)$

### Proof

It is clear using table 2, the equations of lines  $AM'_A$ ,  $BM'_B$  and  $CM'_C$  are

$y(b^2+4Rr) - z(c^2+4Rr) = 0$ ,  $x(a^2+4Rr) - z(c^2+4Rr) = 0$  and  $x(a^2+4Rr) - y(b^2+4Rr) = 0$ .

Now using Proposition 3, to prove that these lines are concurrent, it is enough to prove that

$$\begin{vmatrix} 0 & (b^2 + 4Rr) & -(c^2 + 4Rr) \\ (a^2 + 4Rr) & 0 & -(c^2 + 4Rr) \\ (a^2 + 4Rr) & -(b^2 + 4Rr) & 0 \end{vmatrix} = 0$$

which is true.

They concur at a point  $X(K)$  whose barycentric coordinate is

$((b^2+4Rr)(c^2+4Rr) : (c^2+4Rr)(a^2+4Rr):(a^2+4Rr)(b^2+4Rr))$ . (see figure -3)



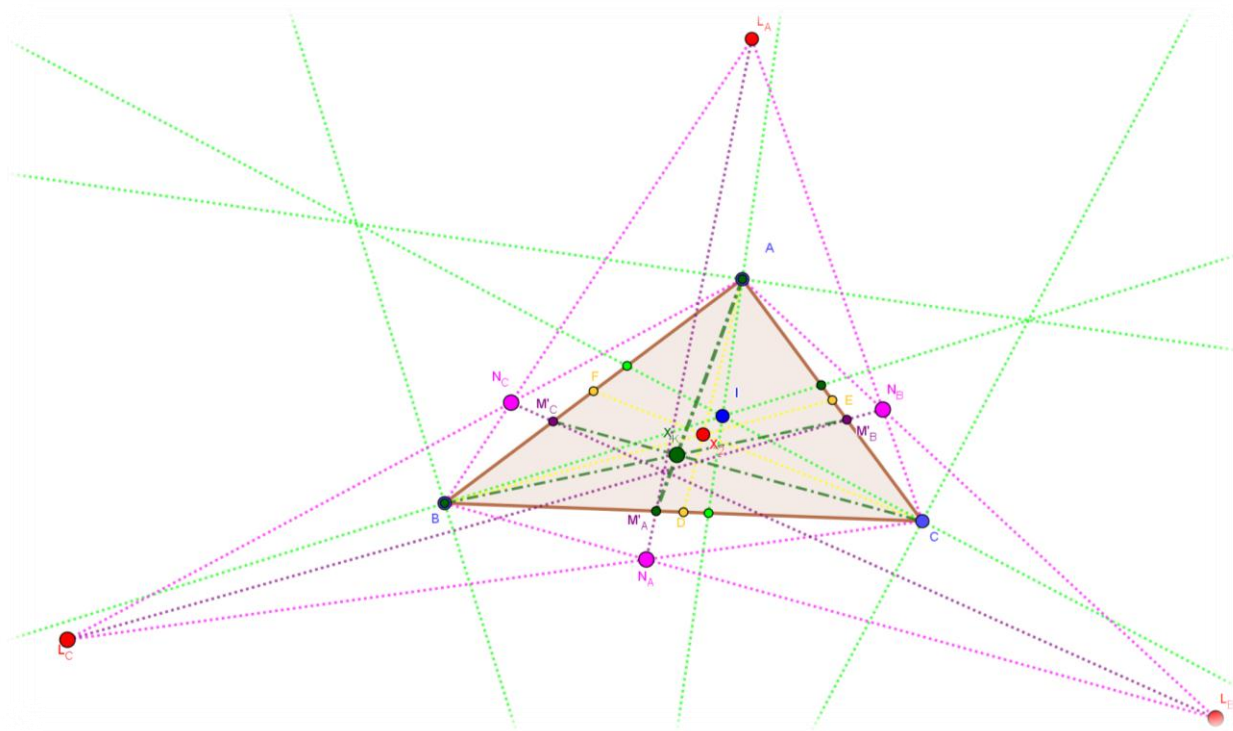


Figure 3.

### Note

The point X(K) is noted as the isogonal conjugate of X(5019) by *Francisco Javier* [5] and as the isotomic conjugate of X(940), the polar conjugate of X(4185) and the trilinear pole of line X(523), X(4391) by *Randy* [6].

### Theorem 4

**The lines  $N_A L'_A$ ,  $N_B L'_B$  and  $N_C L'_C$  are concurrent at X(197)**

### Proof

Using table 2, it is easy to check that the point X(197)  $\{af(A,B,C) : bf(B,C,A) : cf(C,A,B)\}$  satisfies the lines

$$N_A L'_A \{a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0\},$$

$$N_B L'_B \{b^2[z(s-a)-x(s-c)] + y(c-a)[s(c+a)-ca] = 0\}$$

$$\text{and } N_C L'_C \{c^2[x(s-b)-y(s-a)] + z(a-b)[s(a+b)-ab] = 0\}.$$

$$\text{Consider } a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0 \dots\dots(1)$$

$$\text{Replace } x = af(A, B, C), y = bf(B, C, A), z = cf(C, A, B)$$

$$\text{where } f(A, B, C) = a[-a^2 \tan A/2 + b^2 \tan B/2 + c^2 \tan C/2]$$

$$a^2[y(s-c)-z(s-b)] = a^2[b(s-c) f(B, C, A) - c(s-b) f(C, A, B)]$$



$$\begin{aligned}
 &= a^2 \{ s[b f(B, C, A) - c f(C, A, B)] - bc [f(B,C,A) - f(C, A, B)] \} \\
 &= a^2 \{ a^2 \tan A / 2(b-c) (sb+sc-bc) - b^2 \tan B / 2(b-c) (sb+sc-bc) - c^2 \tan C / 2(b-c) (sb+sc-bc) \} \\
 &= -a^2(b-c) (sb+sc-bc) f(A, B, C) = -x(b-c)[s(b+c)-bc]
 \end{aligned}$$

Hence (1) is satisfied by the point X(197)  $\{af(A, B, C) : bf(B, C, A) : cf(C, A, B)\}$

That is the line  $N_A L'_A$  contains X(197).

In a similar manner, we can verify that  $N_B L'_B$  and  $N_C L'_C$  also contains X(197).

So, the lines  $N_A L'_A$ ,  $N_B L'_B$  and  $N_C L'_C$  are concurrent at X(197). (see figure-4)

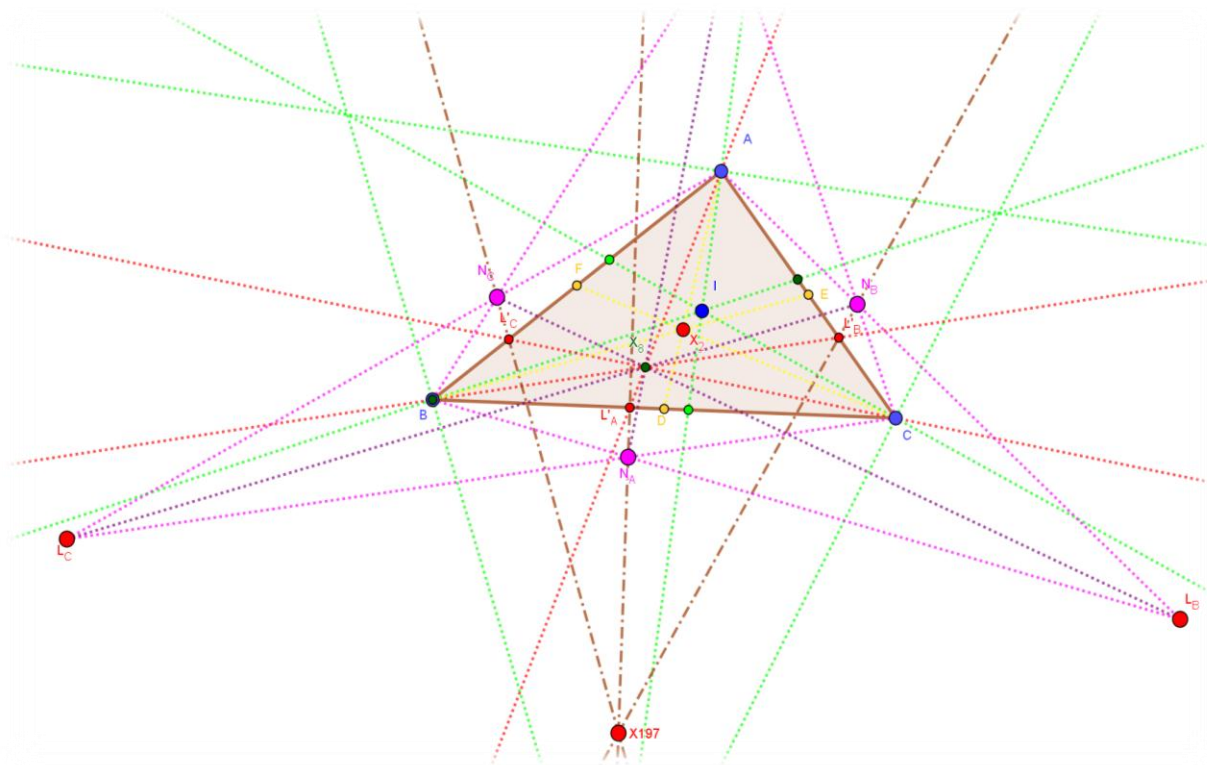


Figure 4.

## Preposition -5

### Carnot's theorem

Suppose a conic  $L$  intersect in the side line  $BC$  at  $X, X'$ ,  $CA$  at  $Y, Y'$  and  $AB$  at  $Z, Z'$  then

$$\left( \frac{BX}{XC} \right) \left( \frac{BX'}{X'C} \right) \left( \frac{CY}{YA} \right) \left( \frac{CY'}{Y'A} \right) \left( \frac{AZ}{ZB} \right) \left( \frac{AZ'}{Z'B} \right) = 1$$

## Preposition -6

If  $X, Y, Z$  are the traces of a point  $P$  and  $X', Y'$  and  $Z'$  are the traces of another point  $Q$  on the sides of  $BC, CA$ , and  $AB$  of the triangle  $ABC$  such that  $P = (u : v : w)$  and  $Q = (u' : v' : w')$ , then using Carnot's theorem there is a conic through six points. The equation of the conic is

$$\sum_{cyclic} \left[ \frac{x^2}{uu'} - \left( \frac{1}{vw'} + \frac{1}{v'w} \right) yz \right] = 0, [4].$$

### Theorem -5

a) There exists conic through  $(D, E, F)$  and  $(N'_A, N'_B, N'_C)$  whose barycentric equation is

$$\sum_{cyclic} \left[ \frac{x^2}{(b+c)^2} - \left( \frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} \right) yz \right] = 0$$

b) There exists conic through  $(N'_A, N'_B, N'_C)$  and  $(L'_A, L'_B, L'_C)$  whose barycentric equation is

$$\sum_{cyclic} \left[ \frac{x^2}{(s-a)(b+c)^2} - \left( \frac{1}{(s-b)(a+b)^2} + \frac{1}{(s-c)(c+a)^2} \right) yz \right] = 0$$

c) There exists conic through  $(D, E, F)$  and  $(L'_A, L'_B, L'_C)$  whose barycentric equation is

$$\sum_{cyclic} \left[ \frac{x^2}{(s-a)} - \left( \frac{1}{(s-c)} + \frac{1}{(s-b)} \right) yz \right] = 0$$

### Proof:

Clearly the triads  $(D, E, F)$ ,  $(N'_A, N'_B, N'_C)$  and  $(L'_A, L'_B, L'_C)$  are the traces of centroid  $\{X(2)\}$   $(1 : 1 : 1)$ , the isogonal conjugate of first Hatzipolakis-Yiu Point  $\{X(594)\}$   $((c+b)^2 : (c+a)^2 : (a+b)^2)$  and Nagel Point  $\{X(8)\}$   $(s-a) : (s-b) : (s-c)$ .

Hence, using Carnot's theorem there exists a conic through the traces of any two of the following three points  $X(2)$ ,  $X(594)$  and  $X(8)$ .

Their corresponding barycentric equations can be calculated using proposition 6.

Using tools similar to those used in this article, the following generalization can be demonstrated.

### Generalization

In a given triangle  $ABC$ ,  $P_a(0 : v : w)$ ,  $P_b(u : 0 : w)$ ,  $P_c(u : v : 0)$  are the traces of an arbitrary point  $P(u : v : w)$  on the sides  $BC, CA, AB$  respectively.  $P_A, P_B$  and  $P_C$  are the reflections of the points  $P_a, P_b, P_c$  with respect to the external angular bisectors of angles  $A, B$  and  $C$ . Let  $Q$  be the isogonal conjugate of  $P$ . Consider the points  $R_A, R_B$  and  $R_C$  as  $R_A = BP_C \cap CP_B$ ,  $R_B = AP_C \cap CP_A$  and  $R_C = BP_A \cap AP_B$ , then

(I) The line segments  $AR_A, BR_B$  and  $CR_C$  concur at  $R$ .

(II) The line segments  $P_AR_A, P_BR_B$  and  $P_CR_C$  concur at  $T$ .

Consider the points  $R'_A, R'_B, R'_C, T'_A, T'_B, T'_C, M'_A, M'_B$  and  $M'_C$  as  $R'_A = AR_A \cap BC$ ,

$$R'_B = BR_B \cap CA, R'_A = CR_A \cap AB, T'_A = AT \cap BC, T'_B = BT \cap CA, T'_C = CT \cap AB,$$

$$S'_A = P_AR_A \cap BC, S'_B = P_BR_B \cap CA \text{ and } S'_C = P_CR_C \cap AB \text{ then}$$

(III) The line segments  $AS'_A, BS'_B$  and  $CS'_C$  are concur at  $S$ .

**(IV) The lines  $R_A T'_A$ ,  $R_B T'_B$  and  $R_C T'_C$  are concur at  $Z$ .**

**(V) The points  $T$ ,  $Q$  and  $R$  are collinear.**

For further study about Nagel's point (X(8)) [2] is referred.

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