Third Order Parallel Splitting Method for Nonhomogeneous Heat Equation with Integral Boundary Conditions

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Abstract

A third order parallel algorithm is proposed in this article to solve one dimensional non-homogenous heat equation with integral boundary conditions. For this purpose, we approximate the space derivative by third order finite difference approximation. This parallel splitting technique is combined with Simpson's 1/3 rule to tackle the nonlocal part of this problem. The algorithm developed here is tested on two model problems. We conclude that our method provides better accuracy due to the availability of real arithmetic.

Keywords: parabolic partial differential equation, non-local boundary conditions, finite difference scheme, integral boundary condition.

1. Introduction

Partial differential equations (PDEs) with initial/boundary conditions (IBC) emerge from the mathematical models of real world problems. The PDEs often appear as mathematical equations relating various quantities and their derivatives, e.g., the movement of a particle in a straight line, the movement of a rocket, heat transition, vibration of a molecule and change in the molecular composition of a substance etc. Each one of these problems is represented by an elliptic, hyperbolic or parabolic partial differential equation (PPDE) and could be homogenous, in one, two or three dimensions with non-local boundary conditions (NLBC) along with initial conditions existing in the prose. In the family of PDEs, one of the most important class is PPDEs with NLBC. This class has been studied by different authors. In real life problems, parabolic equations with integral boundary conditions have a number of applications and sometimes we require only their numerical solutions. Thus PPDEs with NLBC have a considerable impact in fields like electrochemistry, biological and medical sciences and population dynamics [1]. The study of PPDEs with IBC is a well-motivated problem. Whenever it is difficult to develop a mathematical model which contains PPDEs, then nonlocal conditions are widely used in the development of such models pertaining to different physical phenomena [2]. The integral IBCs are the generalized form of discrete but regular IBCs. More specifically, when the boundaries are inaccessible then nonlocal conditions arise during modeling. These partial differential equations are solved by using explicit and Nikolson finite difference (FD) scheme by various authors. Richardson [3] developed the FD technique to solve PDEs. Hartree and Womersley [4] proposed the solution of a PDE with some boundary conditions (BC) using FD approximation. The FD techniques are made by using Taylor expansion. Cannon [5] and Batten [6] independently discuss the development of nonlocal BC PPDEs. Kamynin [7] and Ionkin [8] investigated PPDEs with nonlocal BC for their numerical treatment. Such conditions appear in the modeling of plasma physics, thermal elasticity, heat transmission theory etc [9, 10]. This is the reason PPDEs with NLBC have gained a particular significance in the past and also in the present era. In order to tackle integral conditions which appeared in PDEs many techniques have been proposed in literature and some of these include finite element method, boundary element procedure, spectral schemes, Adomian decomposition approach and the semi-discretization technique [11, 12, 13, 14]. Dehghan [15] proposed three-level explicit finite difference method for the solution of wave 2 equation that merges integral and Neumann condition. Ang [16] developed a numerical technique for solving wave equation with NLBC whose basic assumption is an integro-differential equation and localized interpolating functions. Ramezani and his coworkers [17] introduced another numerical technique by combining FD and the spectral method to obtain numerical solutions of hyperbolic equations subject to NLBC. Bouziani and Benourar [18] studied a mixed PDE which more likely belongs to the class of hyperbolic equations with NLBC in terms of its numerical solution. Cannon and Lin [19, 20] provided a theoretical approach for the solution of PPDEs with NLBC. Dehgan utilized FD schemes [21, 22] and changed Tau method [23] for the solution of a related problem. Now we present the model problem described here. The one dimensional non homogeneous heat equation is given by

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u=\zeta(x,t), \qquad \mathrm{x}\in (0,1), \qquad 0<t<T \tag{1}$$

with the initial condition

$$u(0,t)=F(x), \qquad \mathrm{x}\in(0,1), \qquad 0<t\leq T, \tag{2}$$

and the NBCs

$$\int_0^1 u(x,t)=\sigma_1(t), \qquad 0<t<T, \tag{3}$$

$$\int_0^1 \varphi(x)u(x,t)=\sigma_2(t), \qquad 0<t<T. \tag{4}$$

where ζ(x, t),F(x), σ_1(t), σ_2(t), φ(x) are known and T is a given constant. Different examples of related problems of parabolic equations have been taken into account by many authors [24, 25, 26]. Taj et al. [27] proposed the numerical technique for the solution of PPDEs by utilizing FD scheme and Pade approximation. Rehman et al. [28] proposed the solution of PPDEs with NLBC by combining parallel method with Simpson's 1/3 rule. Following a similar approach, we will utilize Simpson's 1/3 rule for NLBC by combining it with parallel splitting algorithm to obtain a system of z linear ordinary differential equations. The Pade's approximation will be used to approximate matrix exponential function [27].

2. Non Local Boundary Conditions Treatment

Let us take a positive odd integer z ≥ 9 and split spatial range [0, 1] into further z + 1 intervals of length h provided that (z + 1)h = X. Let us also split the open ended variable of time t into subintervals of length l which results into a rectangular mesh having coordinates (x_E, z_n) = (mh, nl)(m = 0, 1, 2, ..., z, z + 1) and (n = 0, 1, 2, 3, ...), provided a region R = [0 < x < 1] × [t > 0] of mesh points and its boundary ∂R involving lines x = 0, x = 1 and t = 0. Consider third order FD scheme given by

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{12}h^2\left\{11u_{j-1}-20u_j+6u_{j+1}+4u_{j+2}-u_{j+3}\right\}-\frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6}+O(h^5),$$

$$\text{as h}\to 0,\ \text{j}=1,2,...,\text{z}-2 \qquad (5)$$

By applying Eq. (5) in equation Eq. (1), we get a compact form as follows

$$\frac{d^2u_j}{dx^2}=\frac{1}{12h^2}\left\{11u_{j-1}-20u_j+6u_{j+1}+4u_{j+2}-u_{j+3}\right\}+\ \zeta_j \qquad (6)$$

This system of ordinary differential equations is valid only for the mesh points (x, t) = (xm, tn) with m = 1, 2...z − 2. Hence we need to develop special FD approximations for remaining mesh points in order to get the same accuracy. So, for m = z−1, z the third order FD schemes with the same accuracy are given by

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{12}h^2\left\{u_{j-3}-6u_{j-2}+26u_{j-1}-40u_j+21u_{j+1}-2u_{j+2}\right\}+O(h^5),\quad \text{as h}\to 0 \qquad (7)$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{12}h^2\left\{2u_{j-4}+11u_{j-3}+24u_{j-2}-14u_{j-1}-10u_j+9u_{j+1}\right\}-\frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6}+O(h^5),\quad \text{as h}\to 0 \qquad (8)$$

By applying Eqs. (6) and (7) in equation Eq. (1), we obtain two differential equations

$$\frac{d^2u_{z-1}}{dx^2}=\frac{1}{12h^2}\left\{u_{j-3}-6u_{j-2}+26u_{j-1}-40u_j+21u_{j+1}-2u_{j+2}\right\}+\ \zeta_{z-1} \tag{9}$$

and

$$\frac{d^2u_z}{dx^2}=\frac{1}{12}h^2\left\{2u_{j-4}+11u_{j-3}+24u_{j-2}-14u_{j-1}-10u_j+9u_{j+1}\right\}+\zeta_{z,} \tag{10}$$

At any time level t = tn, we get a system of z linear ordinary differential equations with z+2 unknowns functions U_0, U_1, ...U_{z+1} which emerge by applying FD approximations to our model problem. The NLBC in Eqs. (4) and Eq. (4) are tackled with the help of Simpson's 1/3 rule in the following way [28],

$$u_0=\frac{h}{3}\left\{\phi_0u_0+4\sum_{j=1}^{\frac{z+1}{2}}\phi_{2j-1}u_{2j-1}+2\sum_{j=1}^{\frac{z+1}{2}-1}\phi_{2j}u_{2j}+\phi_{z+1}u_{z+1}\right\}+\sigma_1(t), \tag{11}$$

and

$$u_1=\frac{h}{3}\left\{\psi_0u_0+4\sum_{j=1}^{\frac{z+1}{2}}\psi_{2j-1}u_{2j-1}+2\sum_{j=1}^{\frac{z+1}{2}-1}\psi_{2j}u_{2j}+\psi_{z+1}u_{z+1}\right\}+\sigma_1(t), \qquad (12$$

In the next few lines we will elaborate the development of system of ordinary liner differential equations for z = 11. Putting z=1,2,3,...,9 in Eq. (6) respectively, we get

$$\frac{du_1}{dt}=\frac{1}{12h^2}\left\{11u_0-(20+12h^2)u_1+6u_2+4u_3-u_4\right\}+\ \zeta_1 \tag{13}$$

$$\frac{du_2}{dt}=\frac{1}{12h^2}\left\{11u_1-(20+12h^2)u_2+6u_3+4u_4-u_5\right\}+\ \zeta_2 \tag{14}$$

$$\frac{du_3}{dt}=\frac{1}{12h^2}\left\{11u_2-(20+12h^2)u_3+6u_4+4u_5-u_6\right\}+\ \zeta_3 \tag{15}$$

$$\frac{du_4}{dt}=\frac{1}{12h^2}\left\{11u_3-(20+12h^2)u_4+6u_5+4u_6-u_7\right\}+\ \zeta_4 \tag{16}$$

$$\frac{du_5}{dt}=\frac{1}{12h^2}\left\{11u(4,t)-(20+12h^2)u_5+6u_6+4u_7-u_8\right\}+\ \zeta_5 \tag{17}$$

$$\frac{du_6}{dt}=\frac{1}{12h^2}\left\{11u(5,t)-(20+12h^2)u_6+6u_7+4u_8-u_9\right\}+\ \zeta_6 \tag{18}$$

$$\frac{du_7}{dt}=\frac{1}{12h^2}\left\{11u(6,t)-(20+12h^2)u_7+6u_8+4u_9-u_{10}\right\}+\ \zeta_7 \tag{19}$$

$$\frac{du_8}{dt}=\frac{1}{12h^2}\left\{11u(7,t)-(20+12h^2)u_8+6u_9+4u_{10}-u_{11}\right\}+\ \zeta_8 \qquad (20)$$

$$\frac{du_9}{dt}=\frac{1}{12h^2}\left\{11u_8-(20+12h^2)u_9+6u_{10}+4u_{11}-u_{12}\right\}+\ \zeta_9 \qquad (21)$$

and putting z = 10 in Eq. (6)

$$\frac{du_{10}}{dt}=\frac{1}{12h^2}\left\{u_7-6u_8+26u_9-(40+12h^2)u_{10}+21u_{11}-2u_{12}\right\}+\ \zeta_{10} \qquad (22)$$

similarly putting z = 11 in Eq. (7)

$$\frac{du_{11}}{dt}=\frac{1}{12h^2}\left\{2u_7-11u_8+24u_9-14u_{10}-(40+12h^2)u(11,t)+9u_{12}\right\}+\ \zeta_{11} \qquad (23)$$

In this way, we can generalize the algorithm and the system of equation can be written in matrix-vector form as

$$\frac{dU(t)}{dt}=AU(t)+v(t), \qquad t>0 \qquad (24)$$

and the initial condition will transform as

$$U(0) \ = \ G \qquad (25)$$

Here $U(t)=[u_1(t),u_2(t),......,u_N(t)]^T$, $X=[\mathrm{g}(\mathrm{x}_1),\mathrm{g}(\mathrm{x}_2),......,\mathrm{g}(\mathrm{x}_N)]^T$, where T denotes transpose and coefficient matrix will transform as

$$A=\frac{1}{12h^2}\begin{bmatrix} \eta_2' & \eta_2' & \eta_3' & \eta_4' & \eta_5' & \eta_6' & \eta_7' & \cdots & \eta_{z-1}' & \eta_z' \\ 11 & \varepsilon & 6 & 4 & -1 & & & \cdots & & \\ & 11 & \varepsilon & 6 & & 4 & & -1 & & \\ \ddots & \ddots & \ddots & \ddots & & \ddots & & \ddots & \ddots & \\ & -1 & 16 & -30 & & 16 & & -1 & & \\ \gamma_1' & \gamma_2' & \gamma_3' & \gamma_4' & & \gamma_5' & & \cdots & \gamma_N' & \\ \delta_1' & \delta_2' & \delta_3' & \delta_4' & & \delta_5' & & \cdots & \delta_N' & \end{bmatrix}_{N\times N} \qquad (26)$$

The quantities in the matrix are given by

$\xi=-20-12h^2$

Where

$\eta_1' = 11E_1 - 20 - 12h,\ \eta_2' = 11E_2 + 6,\ \eta_3' = 11E_3 + 4,\ \ \eta_4' = 11E_4 - 1, \eta_i' = 11E_i,\ for\ i \ge 7,$

$\mu_{z-3} = -D_{z-3} + 11,\ \mu_{z-2} = -D_{z-2} - 20, \mu_{z-1} = -D_{z-1} + 6, \mu_z = -D_z + 4,$

and $\mu_i = -D_i,\ for 1 \le i \le z-4$

$v_{z-4} = -2D_{z-4} + 1,\ v_{z-3} = -2D_{z-3} - 6, v_{z-2} = -2D_{z-2} + 26, v_{z-1} = -2D_{z-1} + 1, v_z = -2D_z + 21$

and $v_i = -2D_i,\ for 1 \le i \le z-5$

$w_{z-4} = 9D_{z-4} + 2,\ w_{z-3} = 9D_{z-3} - 11, w_{z-2} = 9D_{z-2} - 10, w_{z-1} = 9D_{z-1} - 14, w_z = 9D_z - 10 - 12h^2$

and $w_i = 9D_i,\ for 1 \le i \le z-4$

$$E_i = \begin{cases} \dfrac{\frac{4h}{3}(C_4 - C_2 b_i)}{C_1C_4 - C_2C_3} & for\ \text{i=1,3,...,z} \\ \dfrac{\frac{2h}{3}(C_4 - C_2 b_i)}{C_1C_4 - C_2C_3} & \text{for i=2,4,...,z} - 1 \end{cases}$$

and

$$D_i = \begin{cases} \dfrac{\frac{4h}{3}(C_3 - C_1 b_i)}{C_1C_4 - C_2C_3} & for\ \text{i=1,3,...,z} \\ \dfrac{\frac{2h}{3}(C_3 - C_1 b_i)}{C_1C_4 - C_2C_3} & \text{for i=2,4,...,z} - 1 \end{cases}$$

Here

$c_1 = -\dfrac{h}{3},\ c_2 = -\dfrac{h}{3},\ c_3 = -\dfrac{h}{3}\varphi_0$ and $c_4 = -\dfrac{h}{3}\varphi_z,$ also $b_i = b(ih)$

The column matrix contains factors from the functions $\zeta(x,t),\ \sigma_1(t)\ and\ \sigma_2(t)$ and is given as the

$$v(t) = \left[\frac{9l_1}{12h^2} + \zeta_1,\ 0,....0,\ \frac{-l_2}{12h^2} + \zeta_{z-2},\ \frac{-2l_2}{12h^2} + \zeta_z \right]^T$$

where

$$l_1 = \frac{C_2\sigma_2(t) - C_4\sigma_1(\mathrm{t})}{C_1C_4 - C_2C_3}$$

and

$$l_2 = \frac{C_3\sigma_2(t) - C_1\sigma_1(\mathrm{t})}{C_1C_4 - C_2C_3}$$

The solution of system (15) with (2) is given by [27]

$$U(t) = \exp(lA)U(0) + \int_0^{t+l} \exp[(t+l-s)A]v(s)ds \qquad (27)$$

which satisfies the repeat connection

$$U(t+l) = \exp(\mathrm{l}\,A)U(t) + \int_0^{t+l} \exp[(t+l-s)A]v(s)ds \qquad \mathrm{t}=0,\ \ell,\ 2\ell,.... \qquad (28)$$

To surmise the lattice exponential in (2), we utilize the normal approximation for genuine scalar (θ) which is of the shape

$$E_3\theta = \frac{b_0 + b_1\theta + b_2\theta^2}{a_0 - a_1\theta + a_2\theta^2 - a_3\theta^3} \qquad (29)$$

$$a_3 = \sum_{k=0}^{2} (-1)^k \frac{a_k}{(3-k)!} \qquad (30)$$

and

$$b_k = \sum_{i=0}^{K} (-1)^i \frac{a_i}{(K-i)!}, \quad K = 0,1,2,3,4 \qquad (31)$$

Choosing the estimations of parameters a_1, a_2, a_3 as $\frac{91}{20}$, $\frac{481}{120}$ and $\frac{1}{100}$ so that the strategy utilizes only real arithmetic when p and q are factorized into straight components. The fundamental term showing up in condition (2) is approximated as

$$\int_t^{t+l} \exp[(t+l-r)A]v(s)ds = w_1\mathrm{v}(\mathrm{s}_1) + w_2\mathrm{v}(\mathrm{s}_2) + w_3\mathrm{v}(\mathrm{s}_3) \qquad (32)$$

where $\mathrm{s}_1 \neq \mathrm{s}_2 \neq \mathrm{s}_3$ and $\mathrm{W}_1, \mathrm{W}_2$ and W_3 *are matrices.* We have [27]

$$\int_t^{t+l} \exp[(t+l-s)A]s^{k-1}ds = \mathrm{s}^{k-1}{}_j W_j = \mathrm{M}_k,\ \mathrm{k} = 1,2,3, \qquad (33)$$

With

$$\mathrm{M}_k = A^{-1}\{t^{k-1}\exp(lA) - (\mathrm{t+l})^{k-1}\mathrm{I} + (k-1)\,\mathrm{M}_{k-1}\},\ k = 1,2,3 \qquad (34)$$

Taking $s_1 = t,\ s_2 = t + \frac{t}{2},\ s_3 = t + l$. Using $\theta = lA$ in (30) and taking $e^{lA} = \frac{P}{Q}$ we have [27]

$$P = (I - a_1 lA + a_2 l^2 A^2 - a_3 l^3 A^3)^{-1} \qquad (35)$$

$$Q = I + b_1 lA + b_2 l^2 A^2 \qquad (36)$$

$$W_1 = \frac{l}{6} I + (4 - 9a_1 + 12a_2) lAP \qquad (37)$$

where $j = 1, 2, 3$

$$p_{6+j} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{3} \left(1 - \frac{r_j}{r_i}\right)} \times \left\{1 - \left(1 - 3a_1 + 6a_2\right) r_j\right\}$$

where $j = 1, 2, 3$

$$p_{9+j} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{3} \left(1 - \frac{r_j}{r_i}\right)} \times \left\{1 + \left(3 - 9a_1 + 12a_2\right)\left(1 - 3a_1 + 6a_2\right) r^2{}_j\right\}$$

Hence eq. (2) becomes

$$\begin{aligned} U(t+l) = A_1^{-1}\left[p_1 U(t) + \frac{1}{6}\left\{ p_4 v(t) + 4 p_7 v\left(t + \frac{l}{2}\right) + p_{10} v(t+l) \right\} \right] \\ + A_2^{-1}\left[p_2 U(t) + \frac{1}{6}\left\{ p_5 v(t) + 4 p_8 v\left(t + \frac{l}{2}\right) + p_{11} v(t+l) \right\} \right] \\ + A_3^{-1}\left[p_3 U(t) + \frac{1}{6}\left\{ p_6 v(t) + 4 p_9 v\left(t + \frac{l}{2}\right) + p_{12} v(t+l) \right\} \right] \end{aligned}$$

Where $A_i = I - \frac{l}{r_i} A, \qquad i = 1, 2, 3.$

Hence

$$U(t+l) = \sum_{i=1}^{3} yit$$

Where $yi, (i = 1, 2, 3)$ are the solutions of the systems?

3. Applications

Here, we will apply the method develop in the above section to two model problems already found in literature.

Example 1

Assume the heat equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u=2t+t^2+x,\quad x\in(0,1),\quad 0<t<T,\ (38)$$

with initial condition

$$u(0,t)=x,\quad x\in(0,1),\quad 0<t\leq T,\quad (39)$$

and the integral BCs

$$\int_0^1 u(x,t)dx=\frac{1}{2}+t^2,\ 0<t<T \qquad (40)$$

$$\int_0^1 xu(x,t)dx=\frac{1}{3}+\frac{1}{2}t^2,\ \ 0<t<T \qquad (41)$$

Table 1. Error table for example 1 with l=0.00001

$l=10^{-5}$	Exact Solution	Approximate Solution	Relative Error
N=7	1.124980	1.124991	9.6736×10^{-6}
N=9	1.099980	1.099906	9.6751×10^{-6}
N=11	1.083313	1.083324	9.6775×10^{-6}
N=13	1.071409	1.071419	9.6287×10^{-6}
N=15	1.062480	1.062492	9.5876×10^{-6}

We can make sure that the exact solution to this problem is $u(x,t)=x+t^2$ [29]. The numerical solution of the problem is obtained by the method described in the above sections for different values of ℓ= 0.00001, 0.0000001 and z = 7,9,11, 13, 15. The relative error and absolute error are given in Table 1 and Table 2. The results obtained here are very precise which shows that this method is very accurate.

Example 2

Assume the heat equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial t\partial x^2}+u=\left(10-2x\right)e^t,\ \ x\in\left(0,1\right),\quad 0<t<T,\ (42)$$

with initial condition

$$u(0,t)=5-x,\ \ x\in\left(0,1\right),\quad 0<t\le T,\quad (43)$$

and the integral BCs

$$\int_0^1 u(x,t)dx=\frac{9}{2}+e^t,\quad 0<t<T\qquad (44)$$

$$\int_0^1 xu(x,t)dx=\frac{13}{6}e^t,\quad 0<t<T\qquad (45)$$

The exact solution for the issue is $u(u,t)=(5-x)e^t$ [29]. The numerical solution of the problem is obtained by the method described in the above sections for different values of l = 0.00001, 0.0000001 and z = 7, 9, 11, 13, 15. The relative error and absolute error are given in Table 3 and Table 4. The results obtained here are very precise which shows that this method is very accurate.

Table 2. Error table for example 1 with l=0.0000001

$l=10^{-7}$	Exact solution	Approximate Solution	Relative Error
N=7	1.124999	1.125007	6.5984×10^{-6}
N=9	1.099999	1.100009	8.2648×10^{-6}
N=11	1.083333	1.083344	9.6225×10^{-6}
N=13	1.071428	1.071439	1.0778×10^{-6}
N=15	1.062499	1.062512	1.1567×10^{-6}

Table 3. Error table for example 2 with l=0.00001

$l=10^{-5}$	Exact Solution	Approximate Solution	Relative Error
N=7	12.23215	12.23222	6.2741×10^{-6}
N=9	12.23215	12.23223	6.3084×10^{-6}
N=11	12.23215	12.23222	6.3283×10^{-6}
N=13	12.23215	12.23222	6.3376×10^{-6}
N=15	12.23215	12.23222	6.3432×10^{-6}

Table 4. Error table for example 2 with l=0.0000001

$l=10^{-7}$	Exact Solution	Approximate Solution	Relative Error
N=7	11.21291	11.21297	5.6001×10^{-6}
N=9	11.14495	11.14503	6.9319×10^{-6}
N=11	11.09965	11.09974	7.9881×10^{-6}
N=13	11.06729	11.06739	8.9277×10^{-6}
N=15	11.043018	11.043012	9.5369×10^{-6}

4. Conclusion and Discussion

In this work, we have developed a new method for solving nonhomogeneous PPDEs with NLBC. A third order FD scheme is deployed to heat equation to get numerical approximations at grid points. Simpson's 1/3 rule is used to tackle integral boundary conditions which help in the construction of a system z; ordinary differential equation with Z variables. The main role of Simpson's 1/3 rule is the elimination of two additional variables which arise due to NLBC. The developed method is applied to two test problems found in literature and the numerical results obtained here are highly accurate due to the use of real arithmetic only. This technique can be easily coded in serial or parallel computing environment.

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